green square. This completes a square with side lengths c, and area c^2, showing $a^2 + b^2 = c^2$.

Extension

Let c be the length of the hypotenuse of the right triangle.

Area of semicircle with diameter $c = \frac{1}{2}\pi\left(\frac{c}{2}\right)^2 = \frac{\pi c^2}{8}$

Area of right triangle $= \frac{1}{2}ab$

Area of semicircle with diameter $a = \frac{1}{2}\pi\left(\frac{a}{2}\right)^2 = \frac{\pi a^2}{8}$

Area of semicircle with diameter $b = \frac{1}{2}\pi\left(\frac{b}{2}\right)^2 = \frac{\pi b^2}{8}$

Total area of two segments of circle with diameter $c =$ area of semicircle with diameter $c -$ area of triangle $= \frac{\pi c^2}{8} - \frac{1}{2}ab$

Total area of shaded regions = area of semicircle with diameter a + area of semicircle with diameter b − total area of two segments of circle with diameter c:

$$\left(\frac{\pi a^2}{8} + \frac{\pi b^2}{8}\right) - \left(\frac{\pi c^2}{8} - \frac{1}{2}ab\right)$$
$$= \frac{\pi a^2}{8} + \frac{\pi b^2}{8} - \frac{\pi c^2}{8} + \frac{1}{2}ab$$
$$= \frac{\pi}{8}(a^2 + b^2 - c^2) + \frac{1}{2}ab$$

By the Pythagorean Theorem, $a^2 + b^2 = c^2$, so $a^2 + b^2 - c^2 = 0$. Therefore, the area of the shaded region is $\frac{\pi}{8}(0) + \frac{1}{2}ab = \frac{1}{2}ab$, which is the area of the triangle.

EXPLORATION: SHERLOCK HOLMES AND THE FORMS OF VALID REASONING

Developing Mathematical Reasoning

15.2 in. Up 1 in. to get to the top of the vase, around the side of the cylinder a distance that is the hypotenuse of a right triangle with legs 11 in. and 9 in., or approximately 14.2 in.

Total distance is approximately 15.2 inches.

CHAPTER 9 REVIEW

Exercises

1. 20 cm. $15 = 5(3)$ and $25 = 5(5)$. This is a multiple of the Pythagorean triple 3-4-5, so $x = 5(4) = 20$ cm.

2. 10 cm. The altitude from C to $\overline{AB}$ is also a median because C is the vertex angle of an isosceles triangle. This altitude divides the isosceles triangle into two congruent right triangles. Because 5-12-13 is a Pythagorean triple, the shorter leg of one of these right triangles has length 5 cm, so $AB = 2(5) =$ 10 cm.

3. Obtuse. $(70)^2 + (240)^2 = 4{,}900 + 57{,}600 = 62{,}500$, and $(260)^2 = 67{,}600$. Because the sum of the squares of the two shorter sides is less than the square of the longest side, $\angle C$ is obtuse, and therefore the triangle is obtuse.

4. 26 cm. Find the diagonal of the base and then AB, the length of the space diagonal. The diagonal of the base (bottom rectangular face) is $\sqrt{8^2 + 24^2} = \sqrt{64 + 576} = \sqrt{640}$. Then $(AB)^2 = (\sqrt{640})^2 + (6)^2 = 640 + 36 = 676$, so $AB = \sqrt{676} = 26$ cm.

You could also find AB by extending the Pythagorean Theorem to three dimensions, which is similar to using the distance formula in three dimensions: $AB = \sqrt{(24)^2 + (8)^2 + (6)^2} = \sqrt{576 + 64 + 36} = \sqrt{676} = 26$ cm.

5. $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$. Draw the perpendicular segment from U down to the x-axis to form a 30°-60°-90° triangle. The hypotenuse of this triangle is a radius of the circle. The circle has radius 1, so the length of the hypotenuse is also 1. Then, by the 30°-60°-90° Triangle Conjecture, the shorter (vertical) leg has length $\frac{1}{2}(1) = \frac{1}{2}$, and the length of the longer (horizontal) leg is $\frac{1}{2}(\sqrt{3}) = \frac{\sqrt{3}}{2}$. Because U is in Quadrant I, both of its coordinates are positive. Therefore, the coordinates of U are $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$.

6. $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$. Draw the perpendicular from V to the x-axis. Because $225° - 180° = 45°$, the triangle that is formed is a 45°-45°-90° triangle. The hypotenuse of this triangle is a radius of the circle, and the circle has a radius of 1, so the length of the hypotenuse is also 1. Then, by the Isosceles Right Triangle Conjecture, the length of each leg is $\frac{1}{\sqrt{2}}$. Because V is in Quadrant III, both of its coordinates are negative. Therefore, the coordinates of V are $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$.

7. $200\sqrt{3}$ cm² ≈ 346.4 cm². This is a 30°-60°-90° triangle with hypotenuse of length 40 cm. By the 30°-60°-90° Triangle Conjecture, the length of the shorter leg (opposite the 30° angle) is 20 cm, and the length of the longer leg is $20\sqrt{3}$ cm. Then the area of the triangle is $\frac{1}{2}(20)(20\sqrt{3}) = 200\sqrt{3}$ cm² ≈ 346.4 cm².

8. $d = 12\sqrt{2}$ cm ≈ 17.0 cm². The diagonal divides the square into two congruent isosceles right triangles. From the given area, each side of the square has length 12 cm. Because the length of each leg of the isosceles right triangles is 12 cm, by the Isosceles Right Triangle Conjecture, $d = 12\sqrt{2}$ cm ≈ 17.0 cm².

9. 246 cm². Notice that the trapezoid is made up of two right triangles, so the Pythagorean Theorem can be used to find the lengths of its bases. In $\triangle ADC$, $\overline{DA}$ and $\overline{DC}$ are the legs and $\overline{AC}$ is the hypotenuse. Because $12 = 4(3)$ and $20 = 4(5)$, the side lengths in this triangle are a multiple of the Pythagorean triple 3-4-5, so $DC = 4(4) = 16$ cm. In $\triangle ABC$, $\overline{AC}$ and $\overline{BC}$ are the legs and $\overline{AB}$ is the hypotenuse. Because $15 = 5(3)$ and $20 = 5(4)$, the side lengths in this triangle are also a multiple of 3-4-5, so

$AB = 5(5) = 25$ cm. You can now find the area of trapezoid $ABCD$ either by finding the areas of the two triangles separately and adding the results or by applying the Trapezoid Area Conjecture to find the area of the trapezoid directly. Both of these methods are shown here.

Method 1: Use the Triangle Area Conjecture: $A = \frac{1}{2}bh$

Area of trapezoid $ABCD$ = area of $\triangle ADC$ + area of $\triangle ABC = \frac{1}{2}(16)(12) + \frac{1}{2}(20)(15) = 96 + 150 = 246 \text{ cm}^2$

Method 2: Use the Trapezoid Area Conjecture: $A = \frac{1}{2}h(b_1 + b_2)$

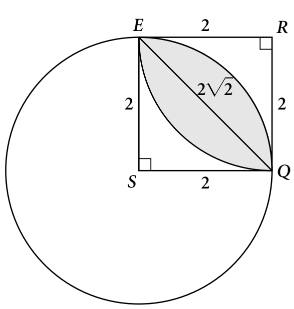
Area of trapezoid $ABCD = \frac{1}{2}(12)(25 + 16) = 246 \text{ cm}^2$

10. $72\pi \text{ in}^2 \approx 226.2 \text{ in}^2$. The diameter of the semicircle is the longer leg of the right triangle. Because 7-24-25 is a Pythagorean triple, the length of this segment is 24 in., so the radius is 12 in. Then the area of the semicircle is $\frac{1}{2}\pi r^2 = \frac{1}{2}\pi(12)^2 = \frac{1}{2}\pi \cdot 144 = 72\pi \text{ in}^2 \approx 226.2 \text{ in}^2$.

11. $24\pi \text{ cm}^2 \approx 75.4 \text{ cm}^2$. The shaded region is a sector of the circle. In order to find its area, you need to know the radius of the circle. Draw $\overline{OT}$ to form two congruent 30°-60°-90° triangles. $m\angle OBT = 90°$ by the Tangent Conjecture, and $m\angle BOT = 60°$ because $\overline{OT}$ bisects the 120° angle, $\angle BOA$. Therefore, in $\triangle BOT$, $\overline{OB}$ is the shorter leg, $\overline{BT}$ is the longer leg, and $\overline{OT}$ is the hypotenuse. To find the radius of the circle, apply the 30°-60°-90° Triangle Conjecture to $\triangle BOT$: $BT = 6\sqrt{3}$ cm, so $OB = 6$ cm. Now find the area of the shaded sector. The angle for this sector is $360° - 120° = 240°$ and the radius is 6 cm.

$$A_{\text{sector}} = \left(\frac{a}{360°}\right)\pi r^2 = \left(\frac{240°}{360°}\right)\pi(6)^2$$
$$= \frac{2}{3}\pi(36) = 24\pi \text{ cm}^2 \approx 75.4 \text{ cm}^2$$

12. $(2\pi - 4) \text{ cm}^2 \approx 2.28 \text{ cm}^2$. Because $SQRE$ is a square, $\triangle ESQ$ is an isosceles triangle with hypotenuse $\overline{QE}$. Because $QE = 2\sqrt{2}$ cm, the length of each side of the square is 2 cm. Use a compass to draw a circle with center S and radius SE.



From the figure, you can see that each half of the shaded region is equal to the area of a segment of the circle in which the central angle is 90°. (The sector is a quarter-circle.) Notice that $\overline{SE}$ and $\overline{SQ}$ are radii of the circle, so $r = 2$ cm. Find the area of the segment.

$$A_{\text{segment}} = A_{\text{sector}} - A_{\text{triangle}}$$
$$= \frac{1}{4}\pi(2)^2 - \frac{1}{2}(2)(2) = (\pi - 2) \text{ cm}^2$$

$$\text{Area of shaded region} = 2A_{\text{segment}}$$
$$= 2(\pi - 2) = (2\pi - 4) \text{ cm}^2$$
$$\approx 2.28 \text{ cm}^2$$

13. 222.8 cm^2. From the figure, you can see that $\overline{BD}$ is both a diagonal of the square and a diameter of the circle. Use r and d to represent the radius and diameter of the circle, respectively. First use the given area of the circle to find the radius: $A = \pi r^2 = 350 \text{ cm}^2$, so $r^2 = \frac{350}{\pi}$ and $r = \sqrt{\frac{350}{\pi}}$ cm. Then $d = 2r = 2\sqrt{\frac{350}{\pi}}$. The diameter is the hypotenuse of isosceles right triangle BAD, so by the Isosceles Right Triangle Conjecture,

$$AB = \frac{2\sqrt{\frac{350}{\pi}}}{\sqrt{2}}$$

$\overline{AB}$ and $\overline{AD}$ are sides of the square. Therefore, the area of square $ABCD$ is

$$\left(\frac{2\sqrt{\frac{350}{\pi}}}{\sqrt{2}}\right)^2 = \frac{4\left(\frac{350}{\pi}\right)}{2}$$
$$= 2\left(\frac{350}{\pi}\right) = \frac{700}{\pi} \approx 222.8 \text{ cm}^2$$

Note: In order to get an accurate result, it is important that you do not round until the final step.

14. Isosceles right triangle. First use the distance formula to find the lengths of the three sides of the triangle.

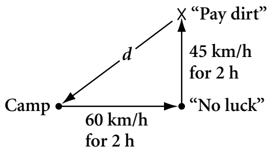
$$AB = \sqrt{(11-3)^2 + (3-5)^2} = \sqrt{8^2 + (-2)^2}$$
$$= \sqrt{64 + 4} = \sqrt{68}$$
$$BC = \sqrt{(8-11)^2 + (8-3)^2} = \sqrt{(-3)^2 + 5^2}$$
$$= \sqrt{9 + 25} = \sqrt{34}$$
$$AC = \sqrt{(8-3)^2 + (8-5)^2} = \sqrt{5^2 + 3^2}$$
$$= \sqrt{25 + 9} = \sqrt{34}$$

Because $BC = AC$, $\triangle ABC$ is isosceles with legs $\overline{AB}$ and $\overline{BC}$. To determine whether this is an isosceles *right* triangle, see whether the side lengths satisfy the Pythagorean Theorem.

$$(BC)^2 + (AC)^2 = \left(\sqrt{34}\right)^2 + \left(\sqrt{34}\right)^2$$
$$= 34 + 34 = 68;$$
$$(AB)^2 = \left(\sqrt{68}\right)^2 = 68$$

Therefore, $\triangle ABC$ is an isosceles right triangle with hypotenuse $\overline{BC}$. (The slopes of $\overline{AC}$ and $\overline{BC}$ could also be used to show that $\angle C$ is a right angle, but since we've already calculated the three lengths, it's faster to just use them.)

15. No. The closest she can come to camp is 10 km. Let d represent the (direct) distance that Sally has traveled from camp. Make a sketch of the situation described in the exercise.



She has traveled $2(60) = 120$ km east and $2(45) = 90$ km north. The distances 120 km and 90 km form the legs of a right triangle in which d is the length of the hypotenuse. Thus, $(120)^2 + (90)^2 = d^2$. Notice that $120 = 30(4)$ and $90 = 30(3)$, so the side lengths in this triangle are a multiple of 3-4-5, and therefore, $d = 30(5) = 150$ km. The distance back to camp is 150 km. Sally's complete trip covers a distance of $120 + 90 + 150 = 360$ km, but she has enough gas to travel only 350 km, so she will fall 10 km short of making it back to camp.

16. No. The 15 cm diagonal is the longer diagonal. A diagonal of a rectangle divides the rectangle into two right triangles, each with the diagonal as a hypotenuse, so you can determine whether the parallelogram is a rectangle by seeing whether the given lengths 8.5 cm, 12 cm, and 15 cm satisfy the Pythagorean Theorem.

$(8.5)^2 + (12)^2 = 72.25 + 144 = 216.25$, while $(15)^2 = 225$, so the parallelogram is not a rectangle. Because $(8.5)^2 + (12)^2 < (15)^2$, the triangle formed by the two sides of the parallelogram and the diagonal of length 15 cm is an obtuse triangle, with the angle between the two sides of the parallelogram an obtuse angle. Every parallelogram that is not a rectangle has two congruent obtuse angles (a pair of opposite angles) and two acute angles (the other pair of opposite angles). The diagonal opposite an obtuse angle of a parallelogram is longer than the diagonal opposite an acute angle, so the 15 cm diagonal must be the longer diagonal of the parallelogram.

17. 1.4 km; 8.5 min. First find the distances that Peter and Paul walk before they stop. Peter walks 2 km/h for 30 min, or $\frac{1}{2}$ h, so he walks 1 km. Paul walks 3 km/h for 20 min, or $\frac{1}{3}$ h, so he also walks 1 km. Because their paths are at right angles to each other, the distance between them after they have both stopped walking is the length of the hypotenuse of an isosceles right triangle in which the length of each leg is 1 km. This distance is $\sqrt{2}$ km $\approx$ 1.4 km. If Peter and Paul start running straight toward each other with *both* of them running at 5 km/h, each of them will travel half this distance, or $\frac{\sqrt{2}}{2}$ km. The time required to travel $\frac{\sqrt{2}}{2}$ km at 5 km/h is

$$\frac{\frac{\sqrt{2}}{2}\text{ km}}{5\text{ km/h}} = \frac{\sqrt{2}}{10}\text{ h}$$

Convert this time to minutes: $\frac{\sqrt{2}}{10}\text{ h} \cdot \frac{60\text{ min}}{1\text{ h}} = 6\sqrt{2}$ min $\approx$ 8.5 min.

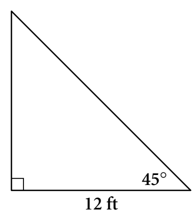
Note: To get an accurate result for the time it will take Peter and Paul to reach each other, work with radicals as shown above, rather than rounding intermediate results. If you use the distance of 1.4 mi to calculate this time, you will get 8.4 min, rather than 8.5 min.

18. Yes. Find the length of the space diagonal of the box. Let d represent the length of the space diagonal. Use the second method shown in the solution for Exercise 4.

$$d = \sqrt{(12)^2 + (16)^2 + (14)^2}$$
$$= \sqrt{144 + 256 + 196} = \sqrt{596} \approx 24.4$$

Thus, the length of the space diagonal of the box is about 24.4 in., so the 24 in. long flute will fit in the box.

19. 29 ft. Make a sketch.



Because the flagpole makes a 45° angle with the ground, the triangle is an isosceles right triangle, so the vertical leg must also be 12 ft long, and the length of the diagonal will be $12\sqrt{2}$ ft $\approx$ 17 ft. The original height of the flagpole is the sum of the lengths of the vertical leg and the hypotenuse, or, to the nearest foot, $12 + 17 = 29$ ft.

20. $\approx$ 45 ft. By the Tangent Conjecture, the angle of the triangle whose vertex is the point of tangency is a right angle. Therefore, the triangle shown in the figure is a right triangle with legs of lengths r and 35 ft, and hypotenuse of length $(r + 12)^2$. Apply the Pythagorean Theorem to this triangle and solve for r.

$$r^2 + (35)^2 = (r + 12)^2$$
$$r^2 + 1225 = r^2 + 24r + 144$$
$$1081 = 24r$$
$$r = \frac{1081}{24} \approx 45$$

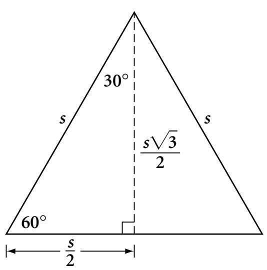
The radius of the tank is approximately 45 ft.

21. Since the distance from the origin to (5, 0) is 5, the radius of the circle is 5. Since the distance from the origin to $(-3, 4)$ is also 5, we can conclude that the point is on the circle.

22. 225π m$^2 \approx 707$ m^2. Draw the radius of the circle to the feet of the diver to form a right triangle. The lengths of the legs of this triangle are 20 m and the radius, while the length of the hypotenuse is 25 m. Because $20 = 5(4)$ and $25 = 5(5)$, the side lengths in this triangle are a multiple of 3-4-5, so the length of the third side is $5(3) = 15$. Therefore, the radius of the circle is 15 m. Find the area of the circular region: $A = \pi r^2 = \pi(15)^2 = 225\pi \approx 707$ m^2.

23. $6\sqrt{3}$ and 18. By the 30°-60°-90° Triangle Conjecture, the length of the shorter leg is half the length of the hypotenuse and the length of the longer leg is the product of the length of the shorter leg and $\sqrt{3}$. In this triangle, the length of the shorter leg is $\frac{1}{2}(12\sqrt{3}) = 6\sqrt{3}$, and the length of the longer leg is $(6\sqrt{3})(\sqrt{3}) = (6)(3) = 18$.

24. 12 m. Let s represent the side length of the equilateral triangle. Draw an altitude (which is also a median and angle bisector) to form two congruent 30°-60°-90° triangles.



In each right triangle, s is the length of the hypotenuse, so by the 30°-60°-90° Triangle Conjecture, the length of the shorter leg is $\frac{s}{2}$ and the length of the longer leg is $\frac{s}{2} \cdot \sqrt{3} = \frac{s\sqrt{3}}{2}$. The longer leg is the altitude of the equilateral triangle. Use the Triangle Area Conjecture and the given area to find the value of s.

$$A = \frac{1}{2}bh$$
$$36\sqrt{3} = \frac{1}{2} \cdot s \cdot \frac{s\sqrt{3}}{2}$$
$$36\sqrt{3} = \frac{s^2\sqrt{3}}{4}$$
$$36 = \frac{s^2}{4}$$
$$s^2 = 144$$
$$s = 12 \text{ m}$$

25. 42. As in the solution for Exercise 24, let s represent the side length of the equilateral triangle. Refer to the sketch in the solution for Exercise 24. In this case, the height of the equilateral triangle, which is the longer leg of each 30°-60°-90° triangle, is $7\sqrt{3}$, so $\frac{s\sqrt{3}}{2} = 7\sqrt{3}$. Solve this equation to find s.

$$\frac{s\sqrt{3}}{2} = 7\sqrt{3}$$
$$s\sqrt{3} = 14\sqrt{3}$$
$$s = 14$$
$$P = 3s = 3(14) = 42$$

The perimeter is 42.

26. Shearing the square into a parallelogram does not change the height or base of the quadrilateral, therefore the areas of the small squares remain the same. When they are added together, they form the large square and thus the areas combine to equal the area of the large square.

27. $(0, 1)$, $(0, -5)$, and $(4, -1)$ are points on the circle because when you substitute each point into the equation, it creates a true statement. Substitute the coordinates of each point for x and y in the equation of the circle to see if the statement is true.

a. $(5 - 1)^2 + (0 + 2)^2 = 4^2 + 2^2 = 20 \neq 10$

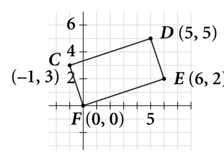
b. $(0 - 1)^2 + (1 + 2)^2 = (-1)^2 + 3^2 = 10 = 10$

c. $(0 - 1)^2 + (-5 + 2)^2 = (-1)^2 + (-3)^2 = 10 = 10$

d. $(4 - 1)^2 + (-1 + 2)^2 = 3^2 + 1^2 = 10 = 10$

28. $A = 25\pi$ square units; $C = 10\pi$ units. The radius of the circle is $\sqrt{25} = 5$. $A = \pi(5)^2 = 25\pi$ square units and $C = 2\pi(5) = 10\pi$ units.

29. Rectangle. To determine the type of special quadrilateral $CDEF$ might be, find the length and slope of each side of the quadrilateral.



$(CD)^2 = (5 - (-1))^2 + (5 - 3)^2 = 6^2 + 2^2 = 36 + 4 = 40$, so $CD = \sqrt{40}$

$(FE)^2 = (6 - 0)^2 + (2 - 0)^2 = 6^2 + 2^2 = 36 + 4 = 40$, so $FE = \sqrt{40}$

$(CF)^2 = (0 - (-1))^2 + (0 - 3)^2 = (-1)^2 + (-3)^2 = 1 + 9 = 10$, so $CF = \sqrt{10}$

$(ED)^2 = (5 - 6)^2 + (5 - 2)^2 = (-1)^2 + (3)^2 = 1 + 9 = 10$, so $ED = \sqrt{10}$

Opposite sides are congruent.

$$\text{Slope of } \overline{CD} = \frac{5-3}{5-(-1)} = \frac{2}{6} = \frac{1}{3}$$

$$\text{Slope of } \overline{FE} = \frac{2-0}{6-0} = \frac{2}{6} = \frac{1}{3}$$